Strong anomalous diffusion: beyond the central limit theorem

Eli Barkai

Bar-Ilan University

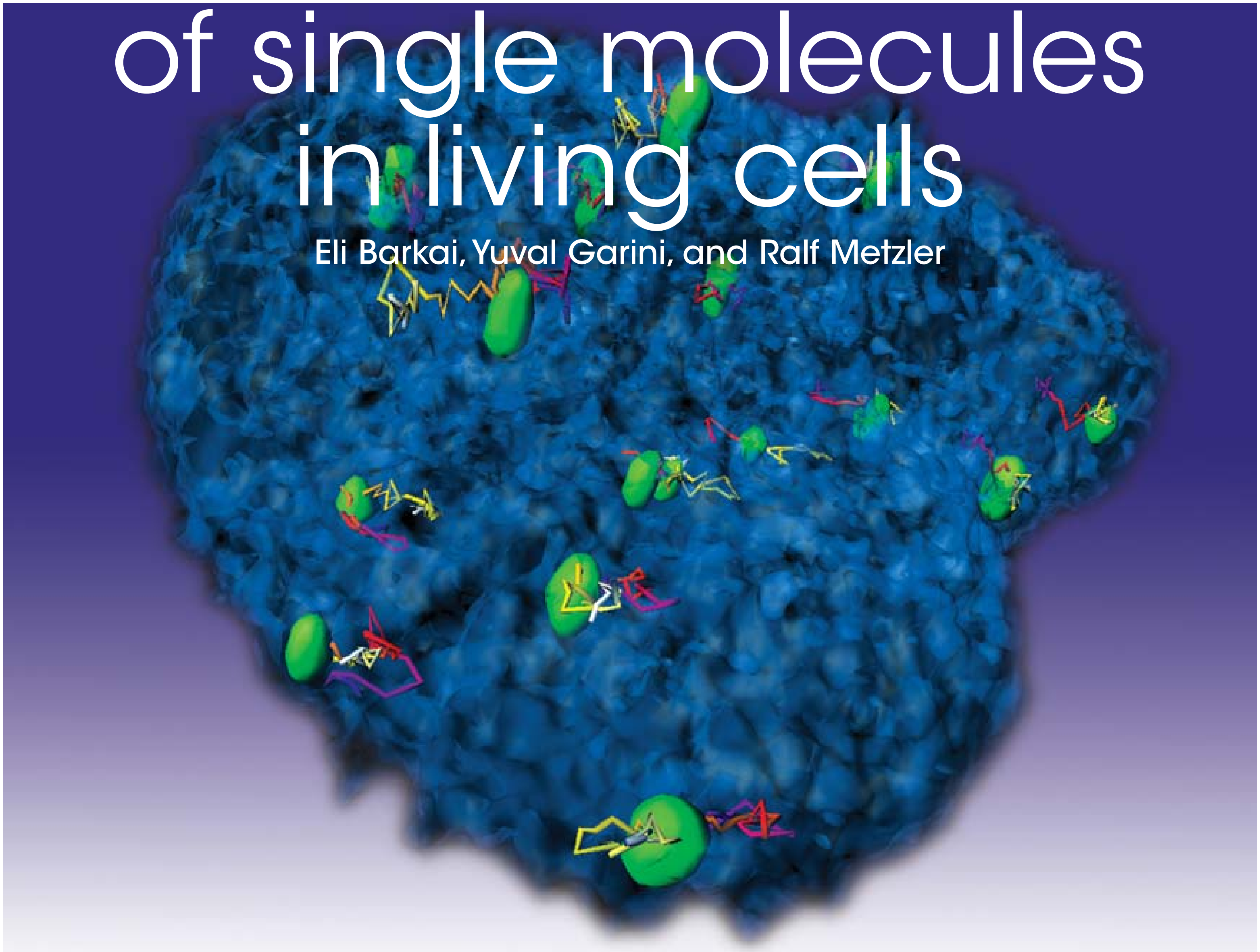
Rebenshtok, Denisov, Hänggi

Phys. Rev. Lett. (2014)

Seoul 2015

STRANGE KINETICS of single molecules in living cells

Eli Barkai, Yuval Garini, and Ralf Metzler



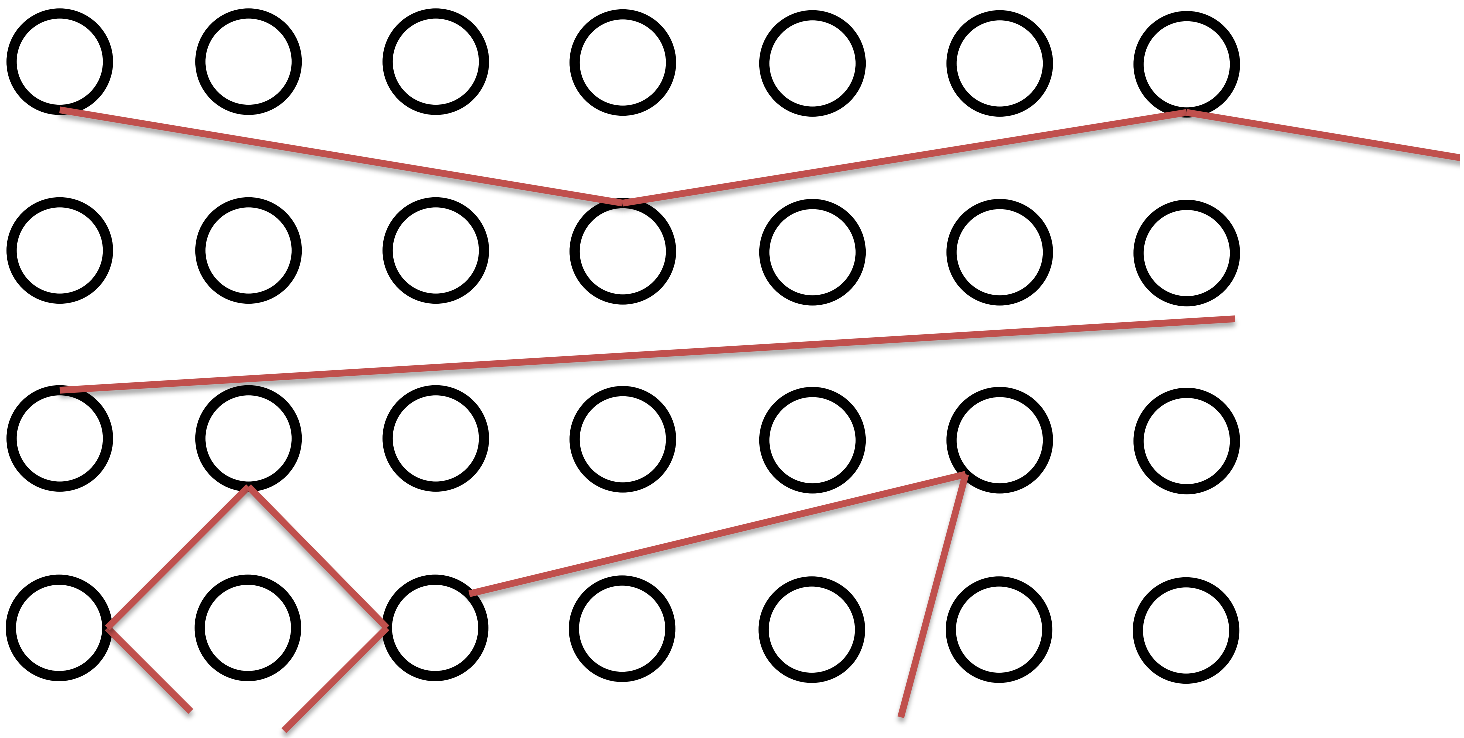
Outline

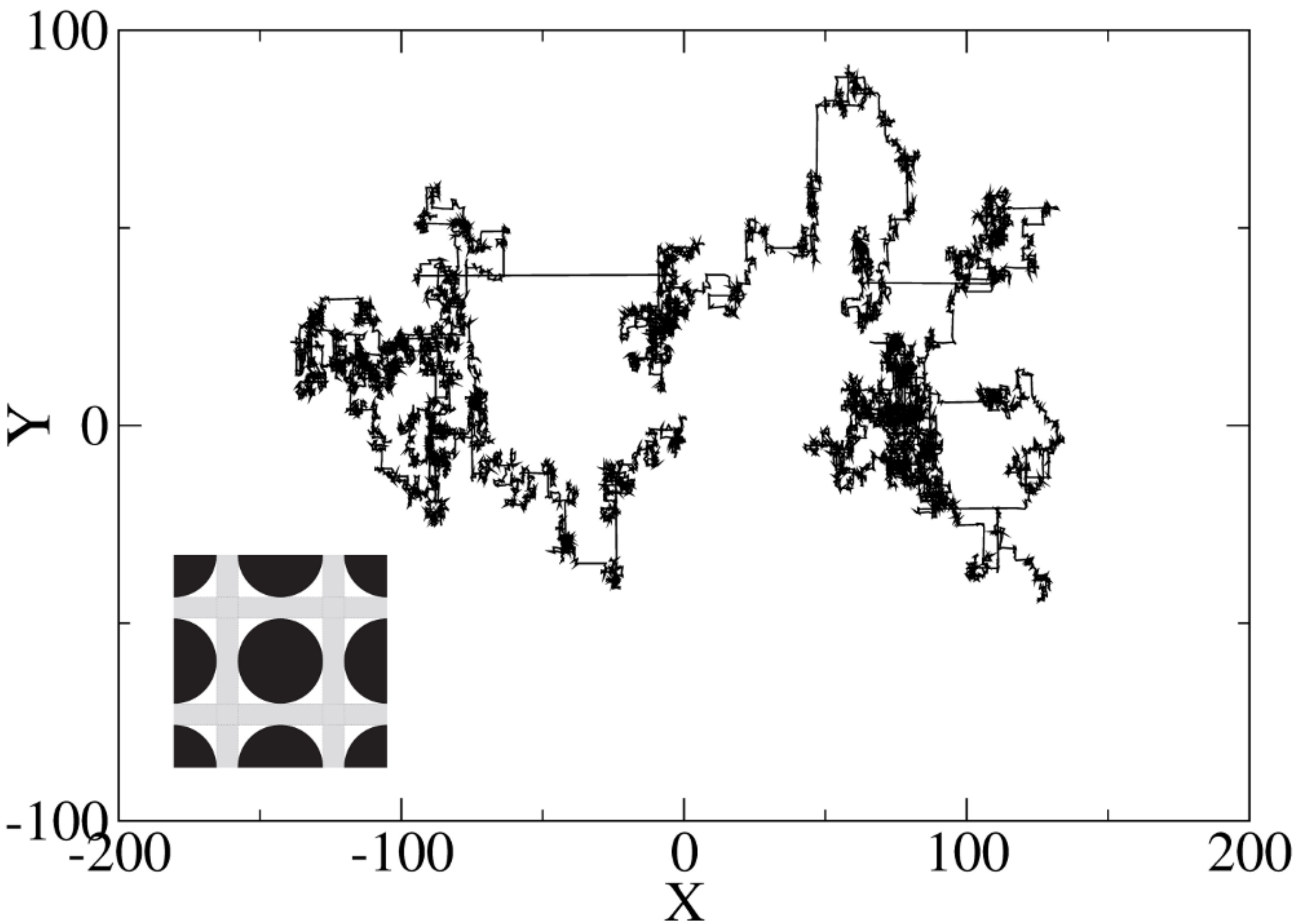
- Strong anomalous diffusion.
- Active transport in live cell (experiment).
- Lévy walk model.
- Non-normalizable infinite densities.

PHYSICAL REVIEW E 77, 036203 (2008)

Problem of transport in billiards with infinite horizon

M. Courbage,[1] M. Edelman,[2] S. M. Saberi Fathi,[1] and G. M. Zaslavsky[2,3]




100
0
-100
Y
-200
-100
0
100
200
X

Strong Anomalous Diffusion

$$\boxed{\langle |x(t)|^q\rangle \sim t^{q\nu(q)}, \qquad \nu(q) \neq \text{const}}$$

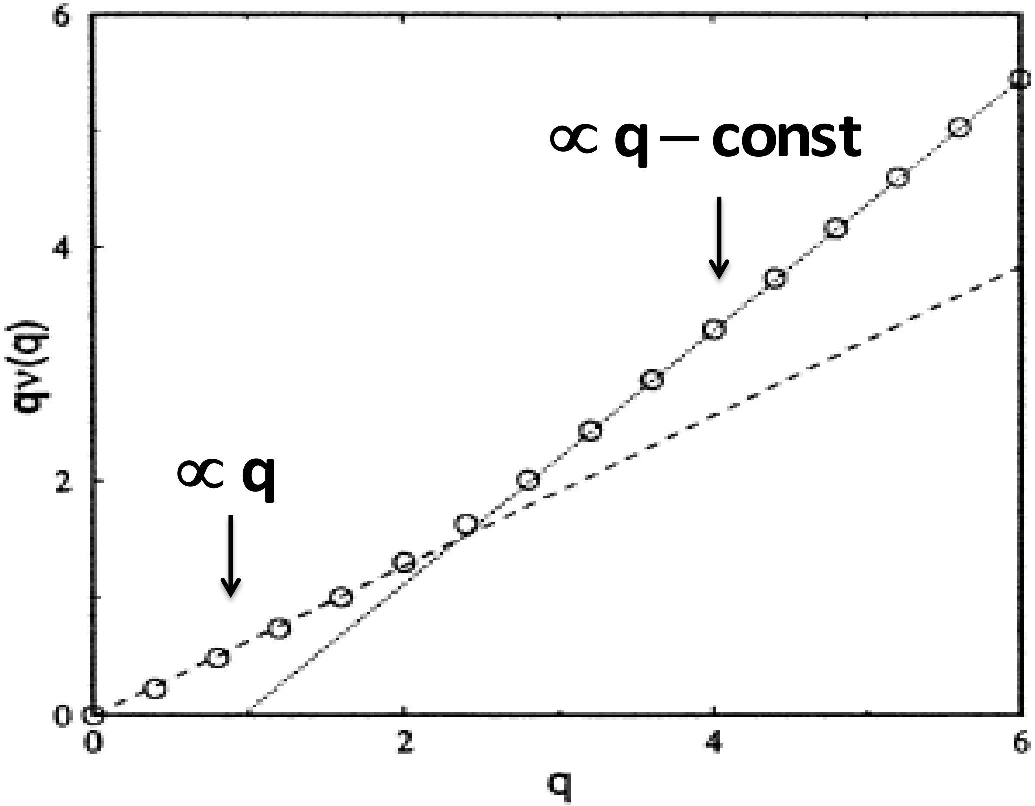
- Brownian motion $\nu(q) = 1/2$.
- Mono-scaling theories are not sufficient or invalid

$$P(x,t) \neq t^{-\nu} f(x/t^{\nu}).$$

Physica D 134 (1999) 75–93

On *strong* anomalous diffusion

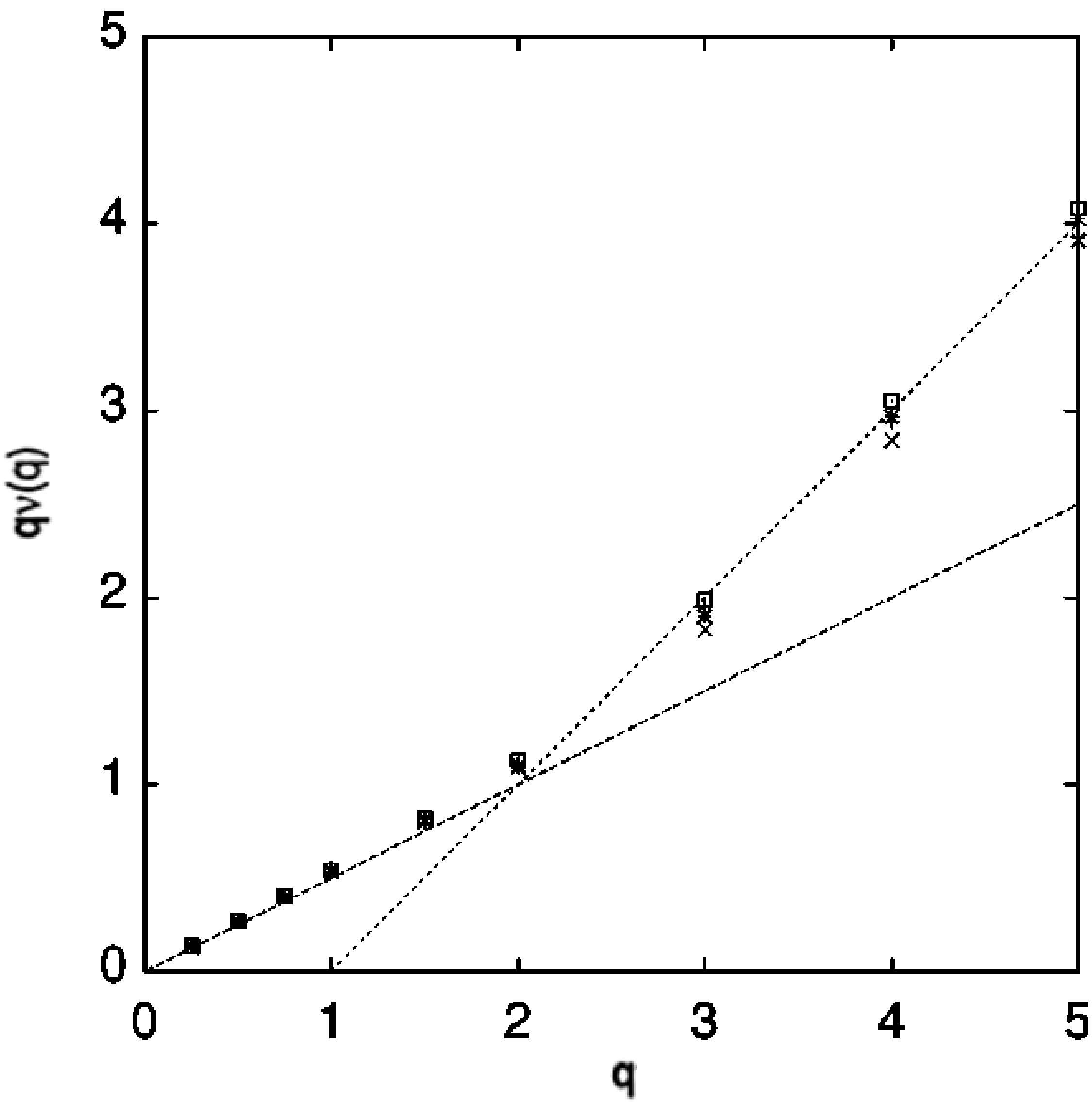
P. Castiglione [a,*], A. Mazzino [b], P. Muratore-Ginanneschi [c], A. Vulpiani [a]



Bi-Linear spectrum, Physical Examples

$$
q\nu(q) \sim \begin{cases} c_1 q & q < q_c \\ c_2 q - c_3 & q > q_c \end{cases}
$$

- Transport in two dimensional incompressible velocity fields (Vulpiani).
- Deterministic transport in intermittent maps (Artuso and Cristadoro).
- Lorentz gas with infinite horizon (AC, Ott, Zaslavsky).
- Diffusion of cold atoms in optical lattices (Barkai, Lutz)
- Active transport in living cells (Weihs)
- Lévy walks a stochastic framework.



qν(q)
q
0
1
2
3
4
5

Questions

- Is dual scaling an asymptotic property?
- Can we describe the diffusive/ballistic packet?
- Go beyond central limit theorem?
- Introduce an infinite density.

Sub-micron particle in live cell (Weihs)

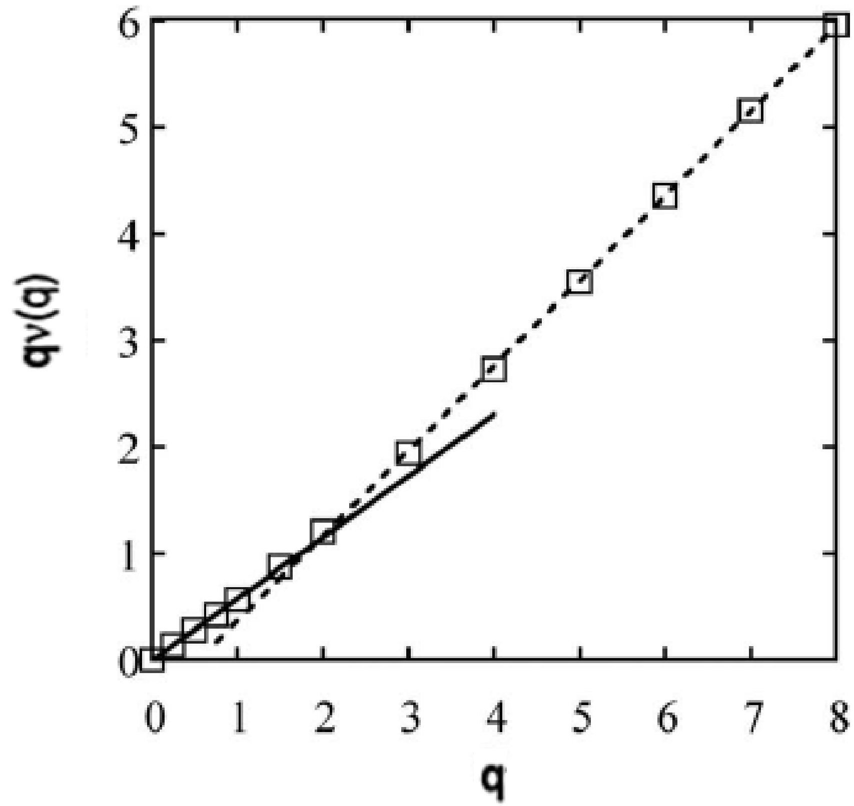
- Single particle tracking reveals super diffusion $\langle X^2(t) \rangle \sim t^{4/3}$.
- Deplete ATP get normal diffusion.
- In this sense the process is called active transport.
- Motion characterized by local confinement separated by active flights.

Scale 50nm

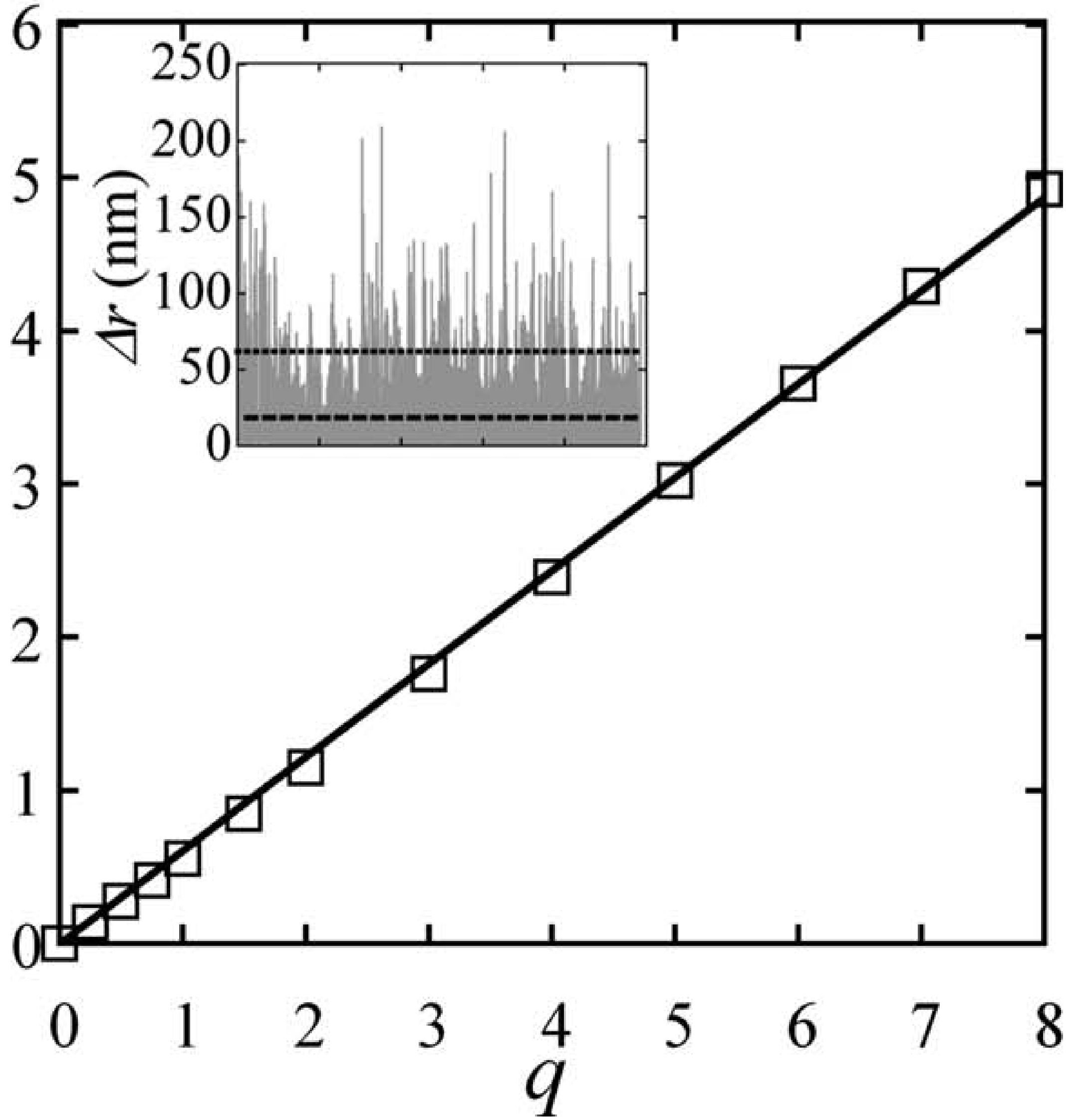
PHYSICAL REVIEW E **81**, 020903(R) (2010)

Experimental evidence of strong anomalous diffusion in living cells

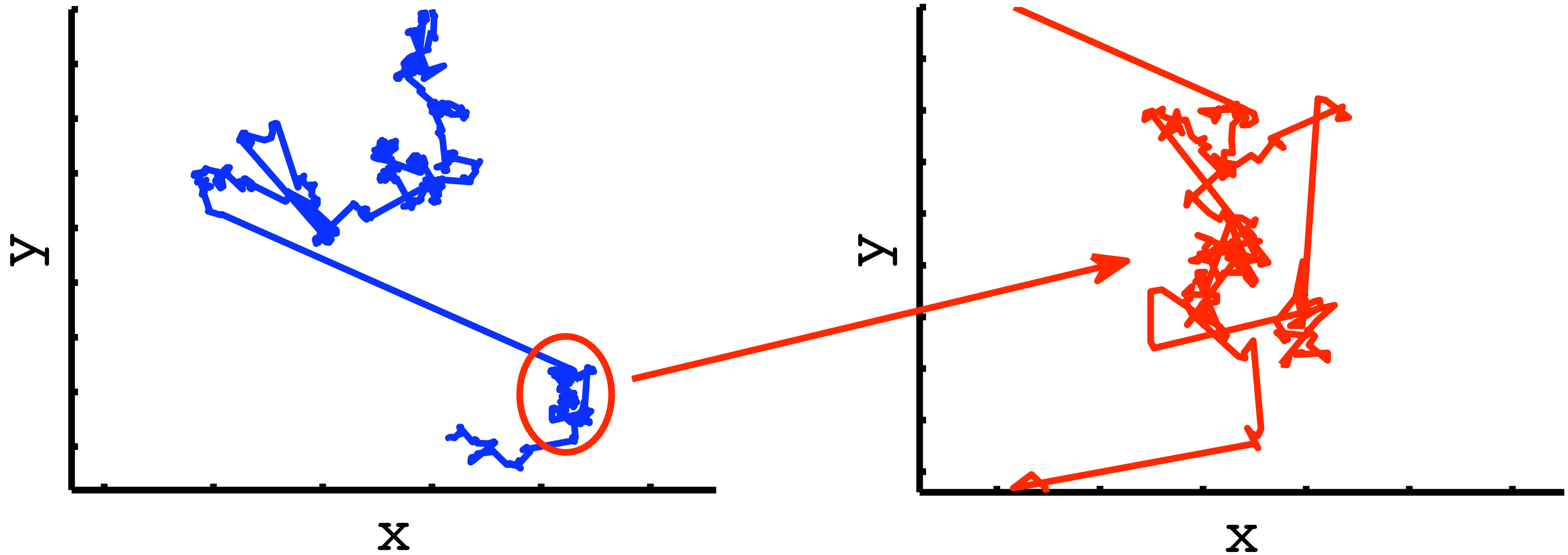
Naama Gal and Daphne Weihs*



qv(q)



Levy walks



- Zarburdaev, Denisov, Klafter, **RMP** (2015)

Model

- Pairs of IID RV (τ_i, v_i).
- PDFs $\psi(\tau)$ and $F(v)$.

$$\boxed{\begin{gathered} t = \sum_{i=1}^{N} \tau_i + \tau^* \\ x = \sum_{i=1}^{N} \chi_i + \chi^* \\ \chi_i = v_i \tau_i \end{gathered}}$$

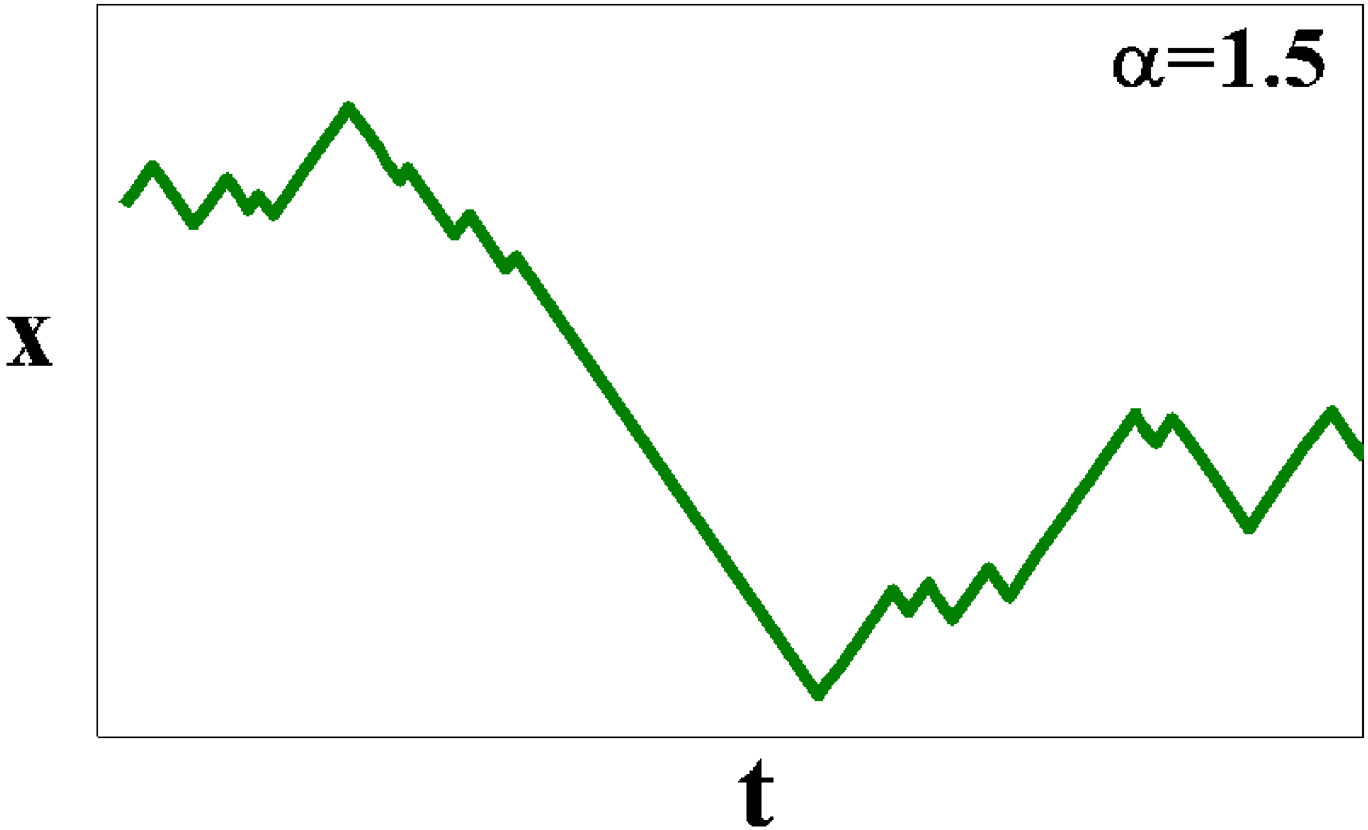
Focus of this talk

- Moments of $F(v)$ are finite and $F(v) = F(-v)$.

$$\boxed{\psi(\tau) \sim \frac{A\tau^{-(1+\alpha)}}{|\Gamma(-\alpha)|} \qquad 1 < \alpha < 2}$$

- For Lorentz gas $\psi(\tau) \sim \tau^{-3}$.

- $\langle \tau \rangle$ finite, variance of the waiting time diverges.


α=1.5
x
t

Central Limit theorem arguments

- $N \simeq t/\langle\tau\rangle$ problem deals with summation of IID RVs?

$$x \simeq \sum_{i=1}^{N} \chi_i$$

- For $1 < \alpha < 2$ apply Lévy's central limit theorem

$$\chi_i = \tau_i v_i.$$

- However competition between Lévy's behavior and ballistic tail, makes the problem interesting.

- Lévy's CLT gives $\langle x^2 \rangle = \infty$, unphysical!

Plan

- Obtain exact expressions for moments $\langle x^n(t) \rangle$
- Use Montroll-Weiss equation and the Faa di Bruno formula.
- Moment generating function (Fourier transform)

$$\boxed{P(k,t) = 1 + \sum_{n=1}^{\infty} \frac{(ik)^n \langle x^n(t) \rangle}{n!}}$$

- Sum the infinite series.
- Take the inverse Fourier transform.
- Get the long time limit of $P(x,t)$? NAIVE.

Let's do it

- For two state model $F(v) = [\delta(v - v_0) + \delta(v + v_0)]/2$

$$\langle x^n(t)\rangle \sim \frac{n}{(n-\alpha)(n+1-\alpha)} \frac{A}{|\Gamma(1-\alpha)|\langle\tau\rangle} (v_0)^n t^{n+1-\alpha}$$

- Summing the series

$$P_A(k,t) \sim 1 + t^{1-\alpha} \frac{A}{|\Gamma(1-\alpha)|\langle\tau\rangle} \tilde{f}_\alpha(ikv_0 t),$$

$$\tilde{f}_\alpha(iy) =$$

$$y^2 \left[\frac{1}{3-\alpha} \ {}_1F_2\left(\frac{3-\alpha}{2}; \frac{3}{2}, \frac{5-\alpha}{2}; \frac{-y^2}{4}\right) - \frac{1}{2-\alpha} \ {}_1F_2\left(1 - \frac{\alpha}{2}; \frac{3}{2}, 2 - \frac{\alpha}{2}; \frac{-y^2}{4}\right)\right]$$

- Take the inverse Fourier transform

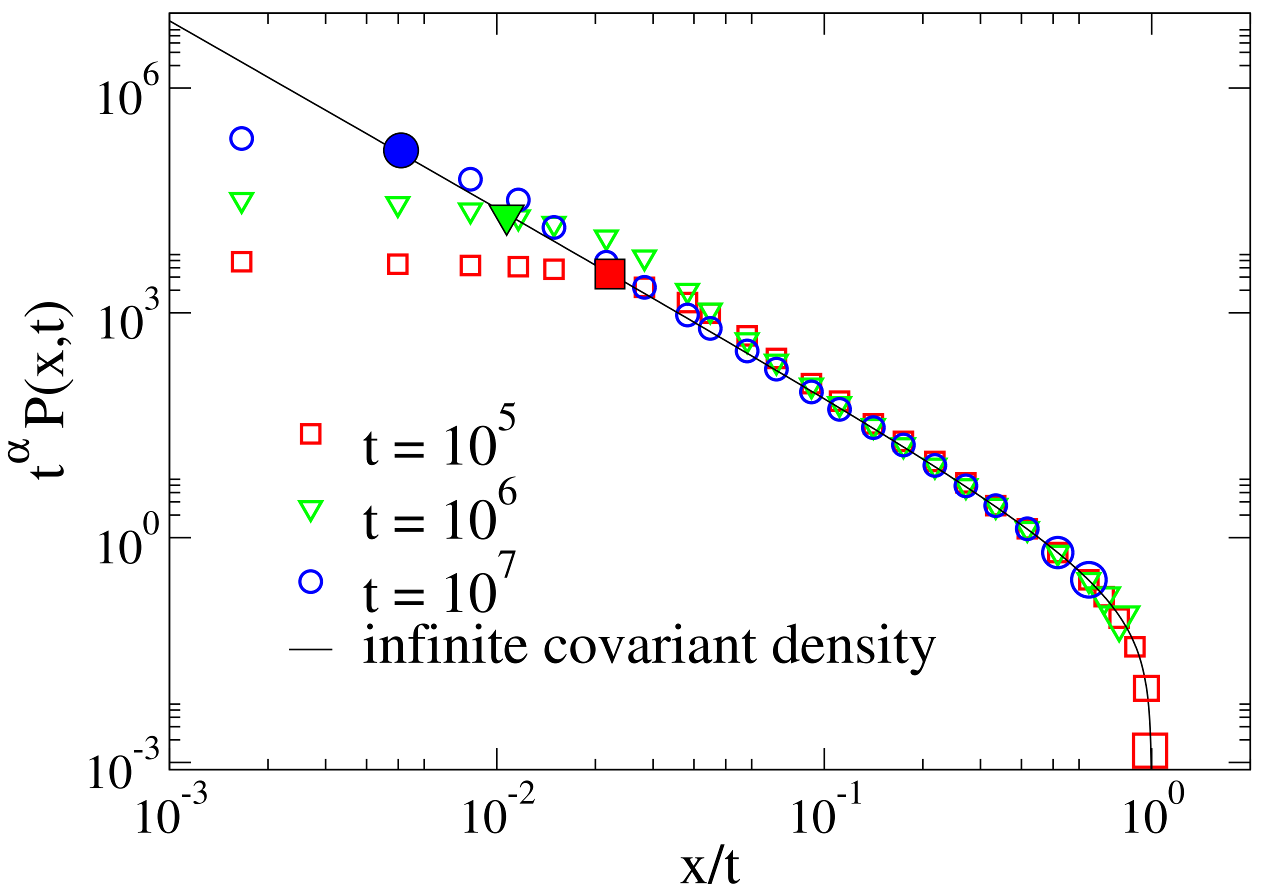
$$\boxed{P_A(x,t) = \frac{\tilde{A}}{t^\alpha}\left|\frac{x}{v_0 t}\right|^{-(1+\alpha)}\left[1 - \left|\frac{\alpha-1}{\alpha}\frac{x}{v_0 t}\right|\right] \quad \text{for} \quad 0 \neq |x| < v_0 t}$$

- Non-normalizable density. TRASH SOLUTION?

- Ballistic x/t scaling.

$$\tilde{A} = A\alpha/2v_0\langle\tau\rangle\,|\Gamma(1-\alpha)|$$

What do simulations say?



Infinite covariant density

- The Infinite covariant density (ICD)

$$\boxed{\lim_{t\to\infty} t^{\alpha} P(x,t) = I_{cd}(\overline{v}) \qquad\qquad \overline{v} \equiv x/t}$$

- For example

$$I_{cd}(\overline{v}) = K_{\alpha} c_{\alpha} |\overline{v}|^{-(1+\alpha)} \left[1 - \frac{\alpha-1}{\alpha}\frac{|\overline{v}|}{v_0}\right]$$

-

Two types of observables:
integrable $(\overline{v}^2)$ and non-integrable $(\overline{v}^0)$
with respect to the ICD.

$K_{\alpha} = A\langle|v|^{\alpha}\rangle|\cos(\pi\alpha/2)|/\langle\tau\rangle \;\; c_{\alpha} = \sin(\pi\alpha/2)\Gamma(1+\alpha)/\pi.$

Fractional diffusion equation?

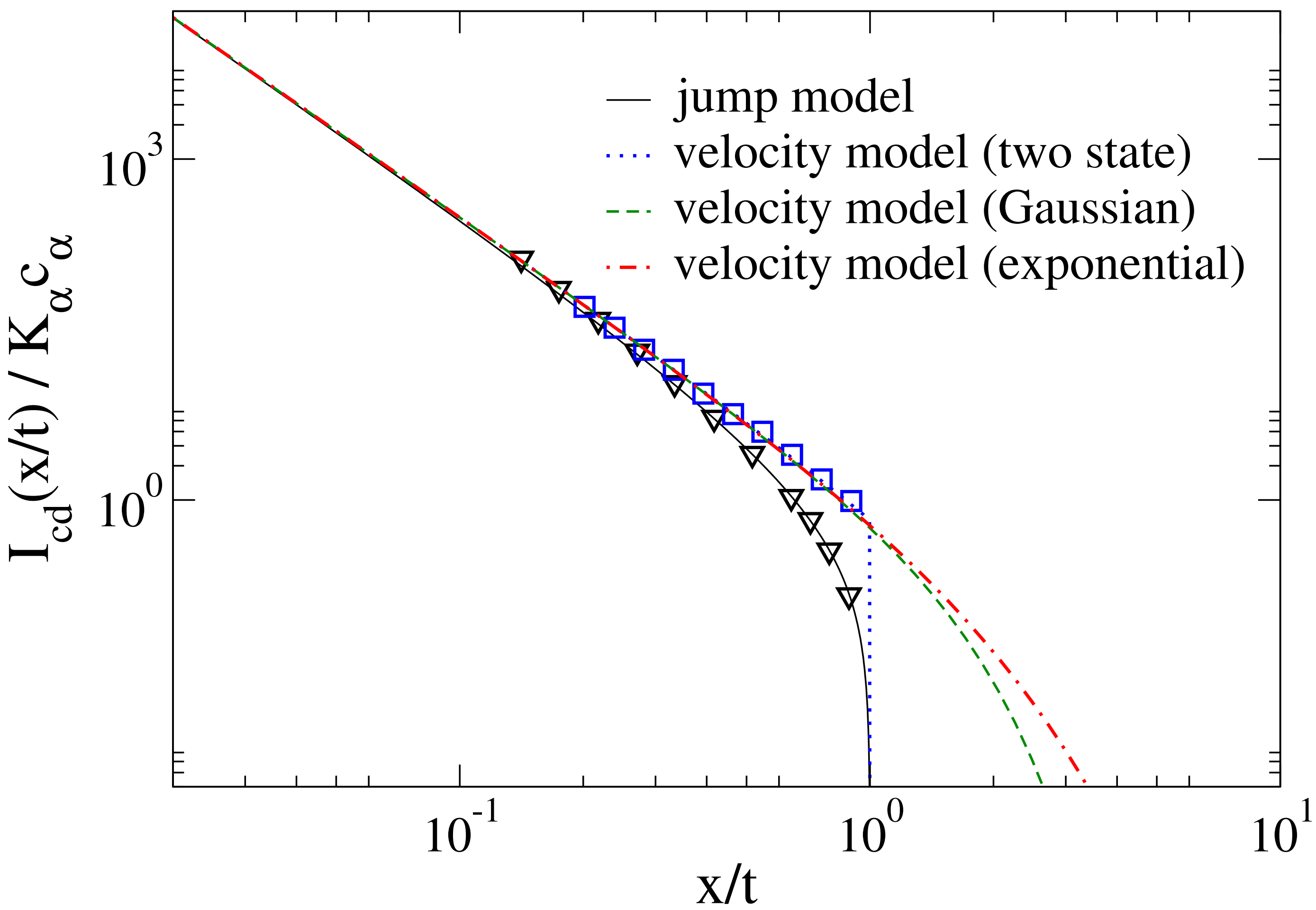
- Lévy's central limit theorem implies that for the *center of the packet*

$$\frac{\partial P_{cen}(x,t)}{\partial t} = K_\alpha \nabla^\alpha P_{cen}(x,t)$$

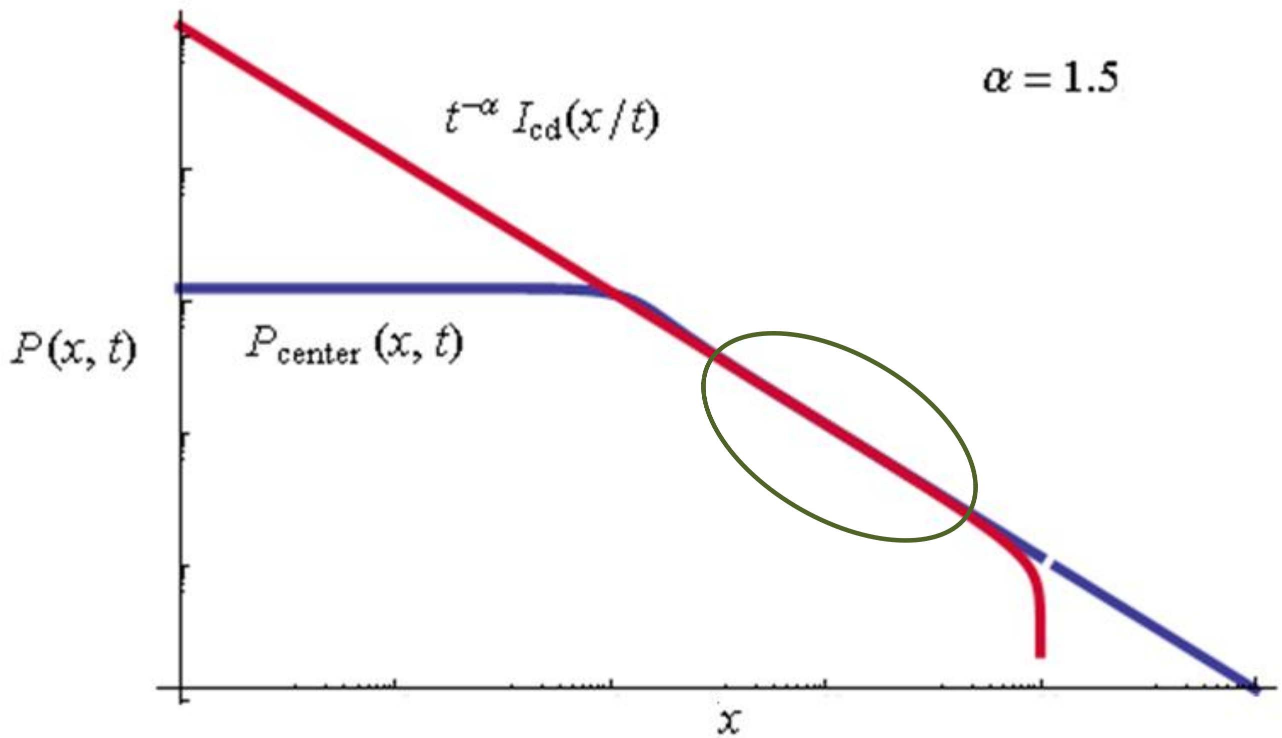
- K_α the anomalous diffusion coefficient can be used to estimate the ICD.

- Observable integrable with respect to Lévy's PDF, i.e., $|x|^q$ and $0 < q < \alpha$, is non integrable with respect to the ICD.

ICD is complementary to the central limit theorem



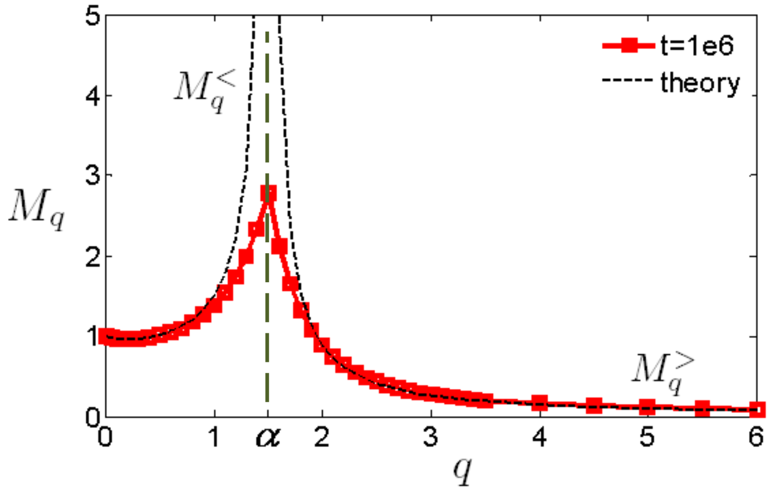
$$I_{cd}(\overline{v}) \sim K_\alpha c_\alpha |\overline{v}|^{-(1+\alpha)} \text{ for } \overline{v} \to 0.$$



$\alpha = 1.5$
$t^{-\alpha} I_{cd}(x/t)$
$P(x,t)$
$P_{center}(x,t)$
x

The Moments

$$\langle |x(t)|^q \rangle = \begin{cases} M_q^{<} \ t^{q/\alpha} & q < \alpha, \\ M_q^{>} \ t^{q+1-\alpha} & q > \alpha. \end{cases}$$



General formula for infinite density

Relation between the ICD and velocity distribution $F(v)$

$$\boxed{\mathcal{I}_{\mathcal{CD}}(\overline{v}) = B\left[\frac{\alpha \mathcal{F}_\alpha(|\overline{v}|)}{|\overline{v}|^{1+\alpha}} - \frac{(\alpha-1)\mathcal{F}_{\alpha-1}(|\overline{v}|)}{|\overline{v}|^{\alpha}}\right]}$$

where

$$\mathcal{F}_\alpha(\overline{v}) = \int_{|\overline{v}|}^{\infty} dv\, v^\alpha F(v)$$

$$B = \frac{\bar{c}_\alpha K_\alpha}{\langle |v|^\alpha \rangle}.$$

Summary

- Dual scaling implies active transport is both quasi ballistic and super diffusive.
- Infinite density is complementary to the central limit theorem.
- Two classes of observables integrable and non-integrable with respect to infinite covariant density.
- Infinite densities describe statistics of a growing number of physical models.

Thanks and Ref.

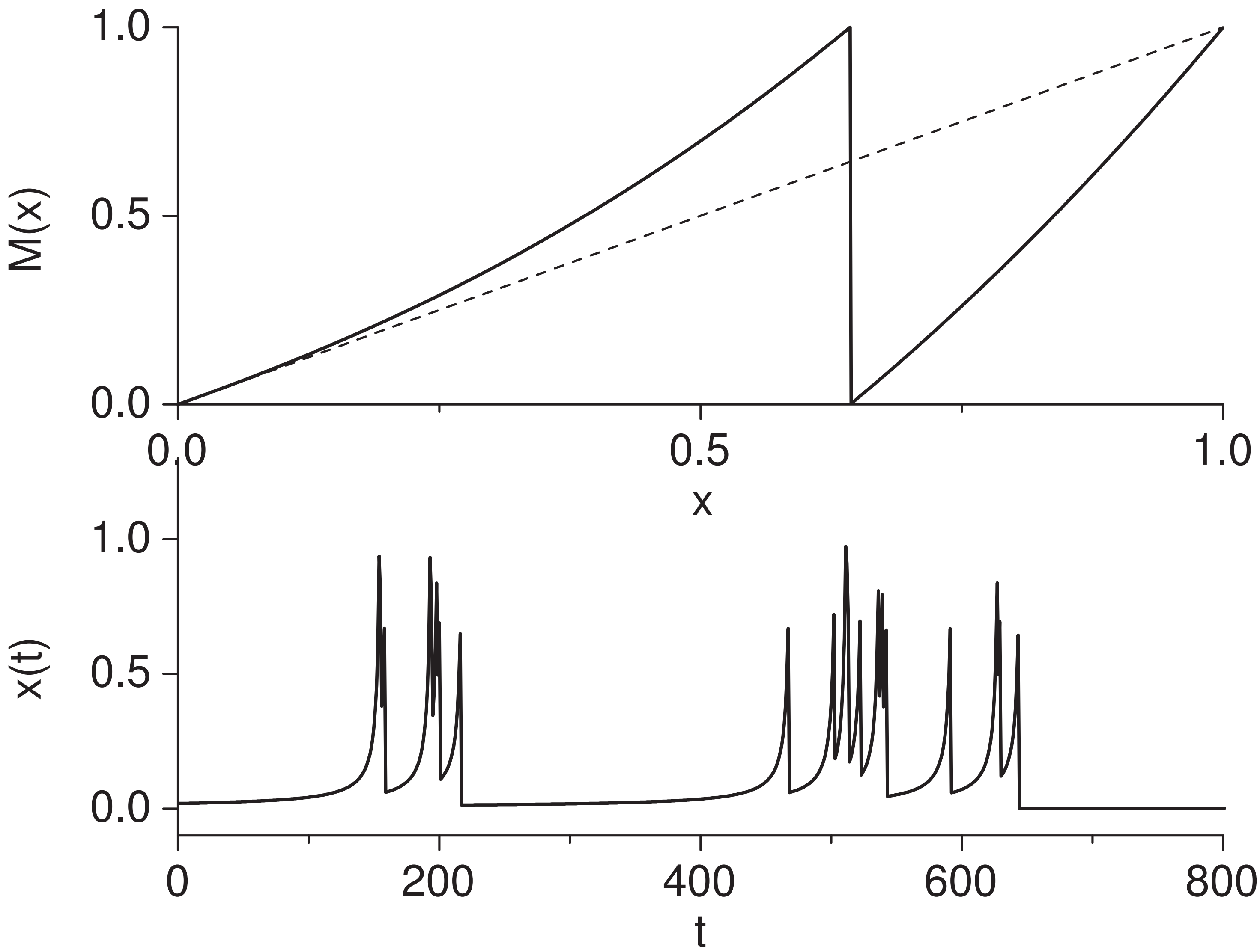
- Rebenshtok, Denisov, Hänggi, EB, **Phys. Rev. Lett. 112**, 110601 (2014)
- ibid **Phys. Rev. E 90**, 062135 (2014).
- Lutz, Renzoni, *Beyond Boltzmann-Gibbs statistical mechanics in optical lattices* Nature Physics
- T. Akimoto and EB *Aging generates regular motions in weakly chaotic systems* **Phys. Rev. E. 87**, 032915 (2013).
- Korabel, EB *Pesin-Type Identity for Intermittent Dynamics with a Zero Lyapunov Exponent* **Phys. Rev. Lett. 102**, 050601 (2009)

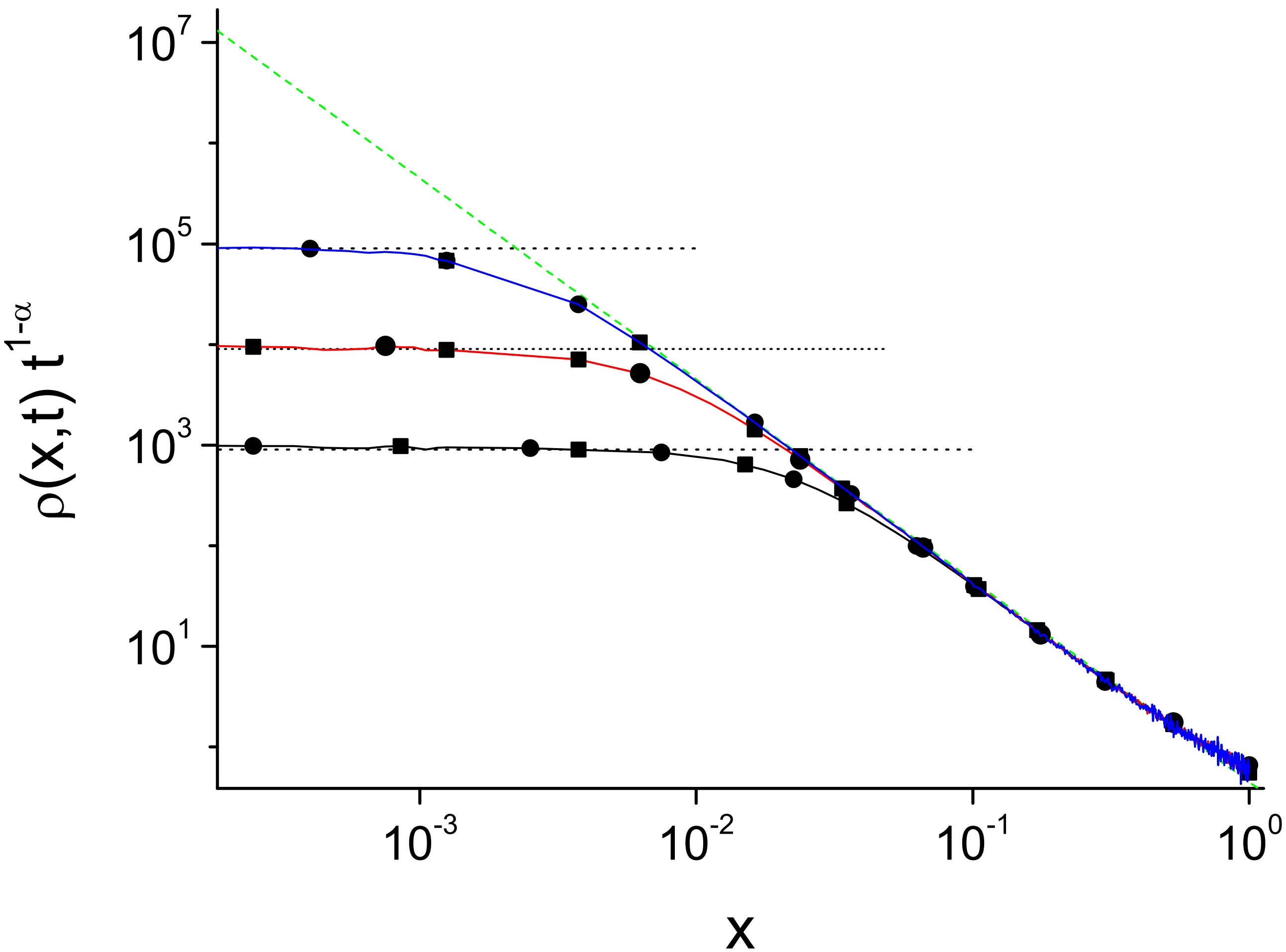
Infinite ergodic theory

- Dynamical systems whose invariant measure is infinite.
- Transformations $x_{t+1} = M(x_t)$ with $0 < x_t < 1$.
- For example the Pomeau-Manneville map:

$$x_{t+1} = x_t + (x_t)^z \ \text{mod} \quad 1, \qquad z \geq 2$$

- Gives $x_0, x_1, ...x_t, ...$ deterministically.
- Exhibits, power law waiting times, $1/f$ noise


M(x)
1.0
0.5
0.0
0.0
0.5
1.0
x
x(t)
1.0
0.5
0.0
0
200
400
600
800
t


10^7
10^5
10^3
10^1
$\rho(x,t)\, t^{1-\alpha}$
10^{-3}
10^{-2}
10^{-1}
10^0
x

Ergodic theory

Time and ensemble averages are identical

$$\lim_{N\to\infty}\sum_{t=0}^{N}\frac{\mathcal{O}[x_t]}{N}\to\int_0^1\mathcal{O}(x)\rho(x)dx.$$

Infinite Ergodic theory

- Non-normalized invariant density, $\rho^{inf}(x) \sim x^{-1/\alpha}$ here $0 < \alpha < 1$ and $\alpha = 1/(z-1)$

$$\int_0^1 \rho^{inf}(x)dx = \infty.$$

- Observable integrable with respect to the infinite density

$$\lim_{N\to\infty} \langle \alpha \sum_{t=0}^{N} \frac{\mathcal{O}[x_t]}{N^\alpha} \rangle \to \int_0^1 \mathcal{O}(x)\rho^{inf}(x)dx.$$

- Application of infinite density concept in physics?

Things to do

- In experiment moment $\langle |x(t)|^q \rangle$ has different time regimes, ballistic then diffusive etc, so emphasize that $q\nu(q)$ is found in long time limit (or show the moments as function of time t).

- Define $P(x, t)$.

- Take fig. from Denisov of trajectory of particle in infinite Lorentz gas. (see above needss to be fixed).

- Emphasize that we take $t \to \infty$ first (in calculation of moments) then obtain infinite series which is summed and inverse Fouriered. Limits do not commute.

- Use fig. frorm PRE (not only $x > 0$).